DUALITY IN ELLIPTIC DIFFERENTIAL GEOMETRY

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I Matrix Methods in Differential Geometry

Let a rectifiable curve $K = \vec{x}(s)$ in Euclidean m-space E^m be given parametrically by the coordinates $x_i(s)$ of its points with respect to a fixed orthonormal coordinate frame $\begin{pmatrix} \mathfrak{J}_1 \\ \vdots \\ \mathfrak{J}_m \end{pmatrix}$. A curve K is of class C^n if each $x_i(s)$, $i = 1, \ldots, m$ has a continuous derivative of order n. We take as parameter s the arc length along curve K measured from a fixed point s_o.

Denoting differentiation with respect to s by a prime (') and differentiation with respect to some other parameter, say t, by a dot ($\dot{}$) we have

<u>lemma 1</u> If on a C^1 curve $\vec{x}(t)$ we take as parameter the arc length $s = \int_{t_o}^{t} [\dot{x}_1(u)^2 + \ldots + \dot{x}_m(u)^2]^{\frac{1}{2}} dt$ then the tangent vector $\vec{x}'$ is a unit vector.

proof:

$$\vec{x}' = \frac{d\vec{x}}{ds} = \frac{d\vec{x}}{dt}\frac{dt}{ds} = \begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_m(t) \end{pmatrix} [\dot{x}_1(t)^2 + \ldots + \dot{x}_m(t)^2]^{-\frac{1}{2}}$$

$$= \frac{\dot{\vec{x}}(t)}{|\dot{\vec{x}}(t)|} \quad \text{a unit vector.}$$

From the fact that two vectors are orthogonal if their inner product vanishes we have

<u>lemma 2</u> The derived vector $\vec{a}'(s)$ of a differentiable vector $\vec{a}(s)$ of constant length is orthogonal to $\vec{a}(s)$.

proof: By hypothesis $\vec{a}(s).\vec{a}(s) = c$ so

$\vec{a}'(s).\vec{a}(s) + \vec{a}(s).\vec{a}'(s) = 2\,\vec{a}(s).\vec{a}'(s) = 0.$

We now define at each point of K an orthonormal coordinate frame (called the moving m-frame) as follows:

Let $\hat{e}_1 = \vec{x}'$ the tangent vector (a unit vector by lemma 1) and let

$$(1)\quad \begin{cases} \hat{e}_1' = k_1\hat{e}_2 \\ \hat{e}_2' = -k_1 e_1 + k_2\hat{e}_3 \\ \vdots \\ \hat{e}_{m-1}' = -k_{m-2}\hat{e}_{m-2} + k_{m-1}\hat{e}_m \\ \hat{e}_m' = -k_{m-1}\hat{e}_{m-1} \end{cases}$$

In the first of equations (1) $\hat{e}_1'$ is a known vector (which by lemma 2 is orthogonal to $\hat{e}_1$) and the convention $k_1 > 0$ determines the direction of $\hat{e}_2$. The requirement that $\hat{e}_2$ be a unit vector determines $k_1 = |\hat{e}_1'|$.

Now assume that the first i-1 of equations (1) have been solved for $k_1, \ldots, k_{i-1}$ and the vectors $\hat{e}_1, \ldots, \hat{e}_i$ are an orthonormal set. Consider the i^{th} equation $\hat{e}_i' = -k_{i-1}\hat{e}_{i-1} + k_i\hat{e}_{i+1}$. $\hat{e}_{i+1}$ must be a unit vector and $k_i > 0$ so $k_i = |\hat{e}_i' + k_{i-1}\hat{e}_{i-1}|$. Forming the dot product of $\hat{e}_{i+1}$ with each of $\hat{e}_1, \ldots, \hat{e}_i$:

$\hat{e}_{i+1}\cdot\hat{e}_i = \frac{1}{k_i}(\hat{e}_i' + k_{i-1}\hat{e}_{i-1})\cdot\hat{e}_i = 0$ since $\hat{e}_{i-1}$ and $\hat{e}_i$ are orthogonal by assumption and $\hat{e}_i'\cdot\hat{e}_i = 0$ by lemma 2. Now $\hat{e}_i\cdot\hat{e}_j = 0$ if $i \neq j$ so $\hat{e}_i'\cdot\hat{e}_j + \hat{e}_i\cdot\hat{e}_j' = 0$ and

$$\hat{e}_{i+1}\cdot\hat{e}_{i-1} = \frac{1}{k_i}(\hat{e}_i' + k_{i-1}\hat{e}_{i-1})\cdot\hat{e}_{i-1} = \frac{1}{k_i}(-\hat{e}_{i-1}'\cdot\hat{e}_i + k_{i-1})$$

$$= \frac{1}{k_i}\left[-\hat{e}_i\cdot(-k_{i-2}\hat{e}_{i-2} + k_{i-1}\hat{e}_i) + k_{i-1}\right]$$

$$= \frac{1}{k_i}(-k_{i-1} + k_{i-1}) = 0 \quad .$$

For $j < i-1$ $\quad \hat{e}_{i+1}\cdot\hat{e}_j = \frac{1}{k_i}\,(\hat{e}_i' + k_{i-1}\hat{e}_{i-1})\cdot\hat{e}_j = \frac{1}{k_i}\,(-\hat{e}_i\cdot\hat{e}_j')$

$$= \frac{-\hat{e}_i}{k_i}\cdot(-k_{j-1}\hat{e}_{j-1} + k_j\hat{e}_{j+1}) = 0$$

so $\hat{e}_{i+1}$ is orthogonal to $\hat{e}_1, \ldots, \hat{e}_i$. Thus the first m-1 of equations (1) do in fact give us an orthonormal m-frame and m-1 uniquely determined positive quantities k_i, $i=1,\ldots,m-1$.

We need only show that the m^{th} equation is consistent with the others. Since $\hat{e}_1, \ldots, \hat{e}_m$ span the space E^m, $\hat{e}_m' = a_1\hat{e}_1 + \ldots + a_{m-1}\hat{e}_{m-1} + a_m\hat{e}_m$. $\hat{e}_m'\cdot\hat{e}_m = 0$ by lemma 2 so $a_m = 0$. $\hat{e}_m\cdot\hat{e}_j = 0 = \hat{e}_m'\cdot\hat{e}_j + \hat{e}_m\cdot\hat{e}_j'$ for $j < m$ so

$$\hat{e}_m'\cdot\hat{e}_j = -\hat{e}_m\cdot\hat{e}_j' = -\hat{e}_m\cdot(-k_{j-1}\hat{e}_{j-1} + k_j\hat{e}_{j+1})$$

$$= \begin{cases} 0 \text{ if } j\ \ m-1 \\ -\hat{e}_m\cdot k_{m-1}\hat{e}_m = -k_{m-1} \text{ if } j=m-1 \end{cases}$$

Hence $a_1 = \ldots = a_{m-2} = 0$ and $a_{m-1} = -k_{m-1}$ giving the desired equation $\hat{e}_m' = -k_{m-1}\hat{e}_{m-1}$.

Since $\begin{pmatrix} \hat{\jmath}_1 \\ \vdots \\ \hat{\jmath}_m \end{pmatrix}$ and $\begin{pmatrix} \hat{e}_1(s) \\ \vdots \\ \hat{e}_m(s) \end{pmatrix}$ are both orthonormal frames, there is an orthogonal transformation $A(s)$ such that

$$\begin{pmatrix} \hat{e}_1(s) \\ \vdots \\ \hat{e}_m(s) \end{pmatrix} = A(s)\begin{pmatrix} \hat{\jmath}_1 \\ \vdots \\ \hat{\jmath}_m \end{pmatrix} .$$

Then $\frac{d}{ds}\begin{pmatrix} \hat{e}_1(s) \\ \vdots \\ \hat{e}_m(s) \end{pmatrix} = A'(s)\begin{pmatrix} \hat{\jmath}_1 \\ \vdots \\ \hat{\jmath}_m \end{pmatrix} = A'(s)\,A^{-1}(s)\begin{pmatrix} \hat{e}_1(s) \\ \vdots \\ \hat{e}_m(s) \end{pmatrix}$.

The product matrix $A'(s)\,A^{-1}(s)$ is called the Frenet matrix and denoted $F(A)$. Concerning the Frenet matrix we prove two lemmas:

lemma 3 $F(AB) = F(A) + A\,F(B)\,A^{-1}$

proof : $F(AB) = (AB)'(AB)^{-1} = (A'B + AB')\,B^{-1}A^{-1}$

$$= A'BB^{-1}A^{-1} + AB'B^{-1}A^{-1}$$

$$= A'A^{-1} + A(B'B^{-1})A^{-1} = F(A) + A\,F(B)\,A^{-1}.$$

lemma 4 If $A(s)$ is orthogonal then $F(A)$ is skew symmetric.

proof: $A^tA = I$ since A is orthogonal.

Differentiating, $A^{t'}A + A^tA' = 0$. Right multiply by A^{-1}, left multiply by $(A^t)^{-1}$, and note that $A^{t'} = A'^t$, $(A^t)^{-1} = (A^{-1})^t$, and $(AB)^t = B^tA^t$ giving

$$A^{t-1}A^{t'}AA^{-1} + A^{t-1}A^tA'A^{-1} = (A'A^{-1})^t + A'A^{-1} = 0,$$

hence $F(A)^t = -F(A)$.

By equations (1) we see

$$\frac{d}{ds}\begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \vdots \\ \hat{e}_{m-1} \\ \hat{e}_m \end{pmatrix} = \begin{pmatrix} 0 & k_1 & 0 & \cdots & & 0 \\ -k_1 & 0 & k_2 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 \cdots & -k_{m-2} & 0 & & k_{m-1} \\ 0 & \cdots & 0 & -k_{m-1} & & 0 \end{pmatrix}\begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \vdots \\ \hat{e}_{m-1} \\ \hat{e}_m \end{pmatrix}$$

which displays $F(A)$ explicitely as a skew symmetric matrix. The $k_i(s)$, $i = 1, \ldots, m-1$ are called generalized curvature quantities.

This is the usual form of the moving m-frame and Frenet matrix. In later sections a slightly different frame, to which this discussion applies, will be used.

Example

For $m = 3$ we write $\hat{e}_1 = \vec{t}$ tangent, $\hat{e}_2 = \vec{n}$ normal, and $\hat{e}_3 = \vec{b}$ binormal. Equations (1) become

$$\begin{aligned}\vec{t}\,' &= \qquad k_1\vec{n} \\ \vec{n}\,' &= -k_1\vec{t} \qquad +k_2\vec{b} \\ \vec{b}\,' &= \qquad -k_2\vec{n}\end{aligned} \qquad \text{and} \quad F(A) = \begin{pmatrix} 0 & k_1(s) & 0 \\ -k_1(s) & 0 & k_2(s) \\ 0 & -k_2(s) & 0 \end{pmatrix}.$$

A mapping $\vec{x} \longrightarrow R\vec{x} + \vec{b}$ is a Euclidean transformation if R is a constant orthogonal matrix and $\vec{b}$ is a constant vector. Two curves are congruent if one is the image of the other under a Euclidean transformation.

lemma 5 If tow curves K_1, K_2 differ only by a translation, $\vec{x}_1(s) = \vec{x}_2(s) + \vec{b}$, they have the same Frenet matrix.

proof: $\vec{t}_1 = \vec{x}_1{}' = \vec{x}_2{}' = \vec{t}_2$. Since the Frenet matrix is determined uniquely by the tangent vector and equations (1), they are the same for the two curves.

lemma 6 If two curves K_1, K_2 differ only by an orthogonal transformation, $\vec{x}_1(s) = R\vec{x}_2(s)$, they have the same Frenet matrix.

proof: In the fixed coordinate system $\begin{pmatrix} \hat{\jmath}_1^* \\ \vdots \\ \hat{\jmath}_m^* \end{pmatrix} = R \begin{pmatrix} \hat{\jmath}_1 \\ \vdots \\ \hat{\jmath}_m \end{pmatrix}$

the moving m-frame of $\vec{x}_1(s)$ is given by

$$\begin{pmatrix} \hat{e}_1 \\ \vdots \\ \hat{e}_m \end{pmatrix} = A \begin{pmatrix} \hat{\jmath}_1 \\ \vdots \\ \hat{\jmath}_m \end{pmatrix} = AR^{-1} \begin{pmatrix} \hat{\jmath}_1^* \\ \vdots \\ \hat{\jmath}_m^* \end{pmatrix} \quad \text{so}$$

$F^*(A) = (AR^{-1})'(AR^{-1})^{-1} = A'R^{-1}RA^{-1} = A'A^{-1} = F(A)$.

We now prove a uniqueness theorem,

Theorem 1 Two curves K_1, K_2 have the same Frenet matrix if and only if they are congruent.

proof: The "if" part follows from lemmas 5 and 6. Let K_1 and K_2 have moving m-frames given by $A(s)$ and $B(s)$

respectively. Since the orthogonal matrices form a group there exists an orthogonal matrix $C(s)$ such that $B(s) = A(s)\,C(s)$. By hypothesis $F(A) = F(B)$ so by lemma 3 $A\,F(C)\,A^{-1} = 0$ so $F(C) = C'C^{-1} = 0$. Thus C is a constant matrix. Hence K_1 can be "rotated" by an orthogonal transformation into K_1^* such that the tangent vector $\vec{t}_1$ of K_1^* is everywhere the same as the tangent vector $\vec{t}_2$ of K_2. Then K_1^* is given by

$$\vec{x}_1^*(s) = \vec{x}_{1o} + \int_o^s \vec{t}_1(u)\,du \qquad \text{and } K_2 \text{ is given by}$$

$$\vec{x}_2(s) = \vec{x}_{2o} + \int_o^s \vec{t}_2(u)\,du = \vec{x}_{2o} + \int_o^s \vec{t}_1(u)\,du$$

so $\vec{x}_1^*(s) - \vec{x}_2(s) = \vec{x}_{1o} - \vec{x}_{2o}$ is constant. Hence K_1^* and K_2 differ by a translation, K_1 and K_2 differ only by a translation and orthogonal transformation.

A matrix $K(s)$ is continuous if each element $k_{ij}(s)$ is continuous, and bounded by b, $|K| < b$, if each element is bounded by b, $|k_{ij}| < b$. If $K(s)$ is continuous on a closed interval, it is bounded on that interval.

We now prove the fundamental theorem of differential geometry.

<u>Theorem 2</u> Given any matrix $K(s)$ continuous for $s_o \leq s \leq s_1$, any s^*, $s_o \leq s^* \leq s_1$, and any orthogonal matrix M and point $\vec{x}(s^*)$, there exists in E^m a unique curve K passing through $x(s^*)$ having $M\begin{pmatrix}\hat{\jmath}_1\\ \vdots \\ \hat{\jmath}_m\end{pmatrix}$ as its m-frame at the point $\vec{x}(s^*)$ and $K(s)$ as its Frenet matrix on some interval about s^*.

proof: First we find a nonsingular matrix $A(s)$ defined on the given interval such that $K(s) = F(A(s))$ and $A(s^*) = I$ where I is the $m \times m$ identity matrix. We solve the matrix differential equation $A'(s)A^{-1}(s) = K(s)$ with initial condition $A(s^*) = I$. We write $A' = KA$ and solve by successive approximation:

$$A_0(s) = I$$

$$A_1(s) = I + \int_{s^*}^{s} K(u)A_0(u)\,du$$

$$\vdots$$

$$A_t(s) = I + \int_{s^*}^{s} K(u)A_{t-1}(u)\,du$$

$$\vdots$$

The continuous $m \times m$ matrix $K(s)$ is bounded on any closed interval about s^*, $|K(s)| \leq b$. Then by induction $|K^t(s)| \leq m^{t-1}b^t$. Also by induction we have

$$(2) \quad |A_t(s) - A_{t-1}(s)| \leq m^{t-1}b^t \frac{|s-s^*|^t}{t!}$$

since $|A_1(s) - A_0(s)| \leq \int_{s^*}^{s} |K(u)|\,du = b|s-s^*|$ and assuming that (2) holds for t,

$$|A_{t+1}(s) - A_t(s)| = \left| \int_{s^*}^{s} K(u)[A_t(u) - A_{t-1}(u)]\,du \right|$$

$$\leq mbm^{t-1}b^t \int_{s^*}^{s} \frac{|u-s^*|^t}{t!}\,du = m^t b^{t+1} \frac{|s-s^*|^{t+1}}{(t+1)!}$$

so (2) holds for all t.

The expression on the right hand side of (2) is the $t+1^{th}$ term in the Taylor series expansion of $\frac{1}{m} e^{mb|s-s^*|}$ which converges uniformly. Therefore $A(s) = \lim_{t\to\infty} A_t(s)$

exists and satisfies the integral equation

$$(3) \qquad A(s) = I + \int_{s^*}^{s} K(u)\, A(u)\, du \ .$$

All the functions $A_t(s)$ are continuous, being integrals of continuous functions. The convergence to $A(s)$ is uniform since $|s-s^*|$ in (2) can be replaced by the larger constant $|s_1-s_0|$. Thus $A(s)$ is continuous and differentiable so $A'(s) = K(s)\, A(s)$ as desired. By construction $A(s^*) = I$. The determinant of $A(s^*)$ is unity, and the determinant of a matrix of continuous functions is itself continuous so there is an interval about s^* in which this determinant is greater than zero and $A(s)$ has an inverse and $A'(s)\, A^{-1}(s) = K(s)$ as desired.

By theorem 1 the unique solution of $B'(s)B^{-1}(s) = K(s)$ with initial condition $B(s^*) = M$ is $B(s) = A(s)\, M$ since $B'(s)B^{-1}(s) = A'(s)\, MM^{-1}A^{-1}(s) = A'(s)A^{-1}(s) = K(s)$.

Now we show that the moving m-frame

$$B(s)\begin{pmatrix} \hat{\jmath}_1 \\ \vdots \\ \hat{\jmath}_m \end{pmatrix}$$

determines a curve K.

We have everywhere in the interval of definition of $B(s)$ the tangent vector $e_1(s)$. Then

$$\vec{x}(s) = \vec{x}(s^*) + \int_{s^*}^{s} e_1(u)\, du \qquad \text{gives the curve K.}$$

K is unique since any other curve having $K(s)$ as its Frenet matrix is congruent to K but the given M and $\vec{x}(s^*)$ single out one curve from the congruence class.

II Connection of Elliptic and Spherical Geometry

Euclids five postulates are

i) any two points determine a unique straight line

ii) a straight line may be produced to any length, still a straight line

iii) a circle of any given radius can be drawn about any given point

iv) all right angles are equal

v) the parallel postulate, equivalent to Playfair's Axiom: through a given point one and only one parallel can be drawn to a given line.

If these are modified as follows

ii)' a straight line is unbounded

iii)' a circle of any given radius less than $\frac{\pi}{2}$ can be drawn about any given point

v)' any two lines in a plane will meet

we have elliptic geometry.

For a rigorous axiomatic development one also needs Hilbert's axioms of incidence, separation, congruence, and continuity (see for example Coxeter [1] pp 20-23 or Wolfe [2] pp 12-16).

It is easily seen that the postulates i), ii)', iii)', iv), v)' hold on a unit sphere in Euclidean space if we identify antipodal points, considering points on the sphere as elliptic points and great circles on the sphere as elliptic lines. The elliptic plane, or sphere with antipodal points identified, is equally well represented by a bundle of lines in Euclidean space,

considering lines of the bundle as elliptic points and planes determined by pairs of lines as elliptic lines.

Either of these representations generalize readily to higher dimensions. To interpret elliptic concepts in terms of Euclidean concepts we simply translate according to one of the following dictionaries (which are restricted here largely to one and two dimensional features):

elliptic m-space S^m	surface of Euclidean (m+1)-dimension sphere	bundle of lines through 0 in Euclidean (m+1)-space
point	pair of antipodal points	line through 0
line	great circle with antipodal points identified	plane through 0
plane	great sphere with antipodal points identified	hyperplane through 0
line segment	arc	angle between two lines
angle	spherical angle	dihedral angle
perpendicular lines	perpendicular arcs	perpendicular planes
triangle	spherical triangle	trihedron
circle	small circle	right circular cone
rotation about a point P	rotation about axis through antipodal points	rotation about line through 0
distance	length of arc	angle between two lines
etc.	etc.	etc.

The representation by Euclidean (m+1)-dimension sphere is the intersection of a unit sphere centered

at O with the bundle of lines through O of the second representation. We will take the surface of a $(m+1)$-dimension Euclidean sphere as a model of elliptic m-space S^m.

A variety in E^m will be any k-dimensional subspace E^k generated by a set $\hat{\imath}_1, \dots, \hat{\imath}_k$ of k m orthonormal vectors concurrent with O; an $(m-1)$-dimensional variety in E^m will be a hyperplane, a 1-dimensional variety in E^m will be a line.

Two varieties E^k and E^l are orthogonal in E^m if every vector in the set $\hat{\imath}_1, \dots, \hat{\imath}_k$ generating E^k is either a linear combination of vectors of the set $\hat{\jmath}_1, \dots, \hat{\jmath}_l$ generating E^l or is orthogonal to all the $\hat{\jmath}$'s.

A k-dimensional s-variety S^k in S^m will be the intersection with the unit sphere in E^{m+1} of a $(k+1)$-dimensional variety in E^{m+1}. A s-hyperplane in S^m is a $(m-1)$-dimensional s-variety, a s-line is a 1-dimensional s-variety. Two s-varieties S^k and S^l are orthogonal in S^m if the corresponding varieties E^{k+1} and E^{l+1} are orthogonal in E^{m+1}.

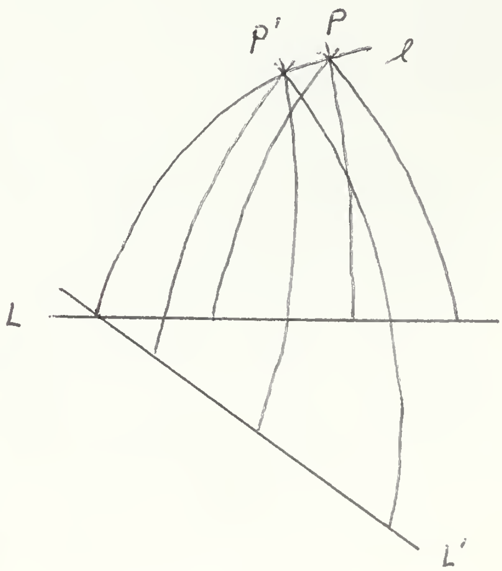
A k-dimensional s-surface (curve, hypersurface, etc.) in S^m is the intersection with the unit sphere in E^{m+1} of a $(k+1)$-dimensional surface in E^{m+1}. A k-dimensional surface in E^m is represented by giving the coordinates of its points parametrically as $x_i(u_1, \dots, u_k)$, $i=1, \dots, m$.

Where confusion will not result we will use simply line, hyperplane, etc. for subspaces of S^m.

Theorem 3 In elliptic space S^m, $m \geq 2$, the lines orthogonal to a hyperplane α are concurrent in a point P at a distance $\frac{\pi}{2}$ from each point of α.

proof: The case m=2 is proven in Sommerville ([3] p 88). This case is easily seen by looking at the ordinary sphere in E^3: all great circles orthogonal to the "equator" meet at the "pole" which is distant $\frac{\pi}{2}$ from the "equator".

Assume the theorem holds for m=k, consider in S^{k+1} the k-dimensional hyperplane α. Consider a (k-1)-dimensional variety L in α. The lines orthogonal to α through L form a k-dimensional variety S^k so by hypothesis these lines are concurrent in a point P distant $\frac{\pi}{2}$ from L, hence distant $\frac{\pi}{2}$ from α. Consider another (k-1)-dimensional variety L' in α distinct from L. The lines orthogonal to α through L' are concurrent in a point P' distant $\frac{\pi}{2}$ from α. But among the lines orthogonal to α there is one, ℓ, common to both L and L'. Both P and P' lie on ℓ distant $\frac{\pi}{2}$ from α so P=P'. Thus all lines orthogonal to α are concurrent in P so the theorem holds for m=k+1. By induction it holds for all m. The diagram illustrates the proof for the case m=3.



The point P is the pole of α and α is the polar of P. A s-hyperplane α in S^m corresponds to an m-dimensional hyperplane in E^{m+1}. The pole of α corresponds to the remaining orthogonal direction in E^{m+1}, i.e. the polar of a point P of S^m represented by $\vec{x}$ is the intersection with the unit sphere in E^{m+1} of the hyperplane orthogonal to $\vec{x}$.

III Duality for Curves in S^2

In equations (1) part I let $\hat{e}_1 = \vec{x}$, a unit vector (since it is on the unit sphere). The arguments of part I apply for any choice of $\hat{e}_1$ so long as it is a unit vector. Then $\vec{e}_1{}' = \vec{x}'$ is the tangent, a unit vector by lemma 1, so $k_1 = 1$ and $\hat{e}_1$ to $\hat{e}_m$ form an orthonormal m-frame with Frenet matrix
$$\begin{pmatrix} 0 & 1 & 0 & \dots & & 0 \\ -1 & 0 & k_1 & \dots & & 0 \\ \vdots & & & & & \\ 0 & \dots & -k_{m-3} & & 0 & k_{m-2} \\ 0 & \dots & 0 & & -k_{m-2} & 0 \end{pmatrix}.$$

In elliptic plane geometry the dual of a curve C is the envelope of the lines which are the polars of points P of C. On the model in E^3 this is the intersection with the unit sphere of the surface which is the envelope of planes dual to the points of C. For point $P=\vec{x}$ on C the dual plane is, substituting plane coordinates for point coordinates, $\vec{\xi}\,\vec{x} = 0$. A point on the envelope must also satisfy $\vec{\xi}\,\vec{x}' = 0$. In E^3 we take $\begin{pmatrix}\vec{x}\\ \vec{x}'\\ \vec{n}\end{pmatrix}$ as the moving m-frame so the curve dual to C is traced by the vector $\vec{\xi}$, $\vec{\xi} = n$ since it is perpendicular to both $\vec{x}$ and $\vec{x}'$. Quantities related to the dual curve C_n will be denoted by a subscript n, quantities related to the curve C by a subscript x.

$$(1) \qquad \frac{d}{ds_x}\begin{pmatrix}\vec{x}\\ \vec{x}'\\ \vec{n}\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & k_x \\ 0 & -k_x & 0 \end{pmatrix}\begin{pmatrix}\vec{x}\\ \vec{x}'\\ \vec{n}\end{pmatrix}$$

where s_x is arc length on C and k_x is the elliptic plane curvature of C.

Denoting the determinant of a matrix whose row vectors are a, b, c by a, b, c we compute x, x', x'' where the prime denotes differentiation with respect to s_x.

$$x, x', x'' = x, x', -x+k_x n$$
$$= x, x', -x + x, x', k_x n$$
$$= k_x \; x, x', n = k_x$$

since a determinant two of whose rows are proportional is zero, a determinant all of whose rows are unit vectors is unity in absolute value and for a right hand system is +1, and x'' is obtained from (1). The curvature k_n of the dual curve is given then by

$$k_n = n, \frac{dn}{ds_n}, \frac{d^2n}{ds_n}$$
$$= n, n' \;, n'' \;\; ^2 + n' \;\; '$$
$$= \;\; ^3 \; n, n', n'' \qquad\qquad \text{where} \quad = \frac{ds_x}{ds_n}$$

and s_n is arc length on the dual curve C_n. From (1)

$$n' = -k_x x' \qquad\qquad \text{so}$$
$$n'' = -k_x' x' - k_x x'' = -k_x' x' + k_x x - k_x^2 n$$

so, dropping terms which are multiples of previous rows,

$$k_n = \;\; ^3 \; n, -k_x x', k_x x = - \;\; ^3 k_x^2 \; n, x', x = k_x^2 \;\; ^3$$

since interchanging adjacent rows changes the sigh of a determinant and x, x', n = 1.

The dual of C_n is traced by perpendicular to both x and n as before so = x', i.e. the dual of the dual of C is C itself. Thus

$$k_x = k_n^2 \; \frac{ds_n}{ds_x}^3 = k_x^4 \; \frac{ds_x}{ds_n}^6 \; \frac{ds_n}{ds_x}^3 \quad .$$

If and k_x are non-zero we can cancel obtaining $1 = k_x^3 \;\; ^3$ so

(2) $$k_x = \frac{ds_n}{ds_x}$$ and

(3) $$k_n = \frac{ds_x}{ds_n} = 1/k_x .$$

Given any smooth curve C in the elliptic plane with curvature k(s) we have

(4) $$\int_C k\, ds = \int_C \frac{ds_n}{ds_x}\, ds_x = \int_C ds_n = L_n ,$$ the length

of the dual curve. This contrasts with the results in Euclidean plane geometry $\int_C k\, ds = 2\pi$ for any simple closed Jordan curve.

The integral (4) is over the curve C on the unit sphere in E^3 and L_n is the length of the dual in spherical geometry. However (4) still holds for the elliptic plane (sphere with antipodal points identified) since the path of integration will be twice around the curve and the integral will be twice the length of the (elliptic) dual. Simular remarks hold for equations (9) and (10) of the next section.

IV Quasi-Duality for Curves in S^3 and S^m

For curve C in elliptic space S^3 we have

$$(1)\qquad \frac{d}{ds_x}\begin{pmatrix}\vec{x}\\ \vec{x}'\\ \vec{t}\\ \vec{n}\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ -1 & 0 & k_{1x} & 0\\ 0 & -k_{1x} & 0 & k_{2x}\\ 0 & 0 & -k_{2x} & 0\end{pmatrix}\begin{pmatrix}\vec{x}\\ \vec{x}'\\ \vec{t}\\ \vec{n}\end{pmatrix}$$

where k_{1x} is the elliptic curvature of C and k_{2x} is the elliptic torsion of C. The dual to a curve in S^3 will be a surface, but inspired by the computation in S^2 and the results yielded by simular computations in S^3 and S^m we define a quasi-dual to curve C in S^3 as the curve C_n traced by $\vec{n}$ and having frame $(\vec{n}, \vec{t}, \vec{x}', \vec{x})$. We compute $[\vec{x}, \vec{x}', \vec{x}'', \vec{x}''']$. From (1)

$$(2)\qquad \begin{cases} \vec{x}'' = -\vec{x} + k_{1x}\vec{t} \\ \vec{x}''' = -\vec{x}' + k_{1x}'\vec{t} + k_{1x}(-k_{1x}\vec{x}' + k_{2x}\vec{n}) \\ \qquad = -(1 + k_{1x}^2)\vec{x}' + k_{1x}'\vec{t} + k_{1x}\vec{n} \end{cases}$$

so omitting terms proportional to previous rows

$$[\vec{x}, \vec{x}', \vec{x}'', \vec{x}'''] = [\vec{x}, \vec{x}', k_{1x}\vec{t}, k_{1x}k_{2x}\vec{n}]$$
$$= k_{1x}^2\, k_{2x}\,[\vec{x}, \vec{x}', \vec{t}, \vec{n}] = k_{1x}^2\, k_{2x}\,.$$

Again the dual of the dual of a curve is the curve itself so

$$k_{1n}^2\, k_{2n} = \left[\vec{n}, \frac{d\vec{n}}{ds_n}, \frac{d^2\vec{n}}{ds_n^2}, \frac{d^3\vec{n}}{ds_n^3}\right]$$
$$= [\vec{n}, \vec{n}'\varphi, \vec{n}''\varphi^2 + \vec{n}'\varphi'\varphi,$$
$$\vec{n}'''\varphi^3 + 3\vec{n}''\varphi'\varphi^2 + \vec{n}'(\varphi''\varphi^2 + \varphi'^2\varphi)]$$
$$= [\vec{n}, \vec{n}'\varphi, \vec{n}''\varphi^2, \vec{n}'''\varphi^3]$$
$$= \varphi^6[\vec{n}, \vec{n}', \vec{n}'', \vec{n}''']$$

where $\varphi = \frac{ds_x}{ds_n}$ as before. We compute

$$(3)\quad \begin{cases} \vec{n}' = -k_{2x}\vec{t} \\ \vec{n}'' = -k_{2x}'\vec{t} - k_{2x}\vec{t}' = -k_{2x}'\vec{t} - k_{2x}(-k_{1x}\vec{x}' + k_{2x}\vec{n}) \\ \vec{n}''' = -k_{2x}''\vec{t} - k_{2x}'(-k_{1x}\vec{x}'+k_{2x}\vec{n}) - k_{2x}'(-k_{1x}\vec{x}'+k_{2x}\vec{n}) \\ \qquad -k_{2x}[-k_{1x}'\vec{x}' - k_{1x}(-\vec{x}+k_{1x}\vec{t})+k_{2x}'\vec{n}+k_{2x}(-k_{2x}\vec{t})] \end{cases}$$

omitting the unnecessary terms

$$[\vec{n}, \vec{n}', \vec{n}'', \vec{n}'''] = [\vec{n}, -k_{2x}\vec{t}, k_{2x}k_{1x}\vec{x}', -k_{2x}k_{1x}\vec{x}]$$
$$= k_{1x}^{2}\, k_{2x}^{3}[\vec{n}, \vec{t}, \vec{x}', \vec{x}] = k_{1x}^{2}\, k_{2x}^{3} \quad \text{so}$$

$$(4)\qquad k_{1n}^{2}\, k_{2n} = k_{1x}^{2}\, k_{2x}^{3}\varphi^{6} .$$

Now using equations (2)

$$[\vec{x}, \vec{x}', \vec{x}'', \vec{n}] = [\vec{x}, \vec{x}', k_{1x}\vec{t}, \vec{n}] = k_{1x}$$

so, using equations (2) and (3),

$$k_{1n} = \left[\vec{n}, \frac{d\vec{n}}{ds_n}, \frac{d^2\vec{n}}{ds_n^2}, \vec{x}\right] = \varphi^3[\vec{n}, \vec{n}', \vec{n}'', \vec{x}]$$
$$= \varphi^3[\vec{n}, -k_{2x}\vec{t}, k_{2x}k_{1x}\vec{x}', \vec{x}]$$
$$= -\varphi^3 k_{1x}k_{2x}^{2}[\vec{n}, \vec{t}, \vec{x}', \vec{x}],$$

$$(5)\qquad k_{1n} = -k_{1x}k_{2x}^{2}\varphi^{3} .$$

From (4) and (5) we have $k_{1x}^{2}\, k_{2x}^{4}\varphi^{6}k_{2n} = k_{1x}^{2}\, k_{2x}^{3}\varphi^{6}$ or, if k_{1x}, k_{2x}, and φ are not zero,

$$(6)\qquad k_{2n} = 1/k_{2x} .$$

Now $\frac{d\vec{n}}{ds_n} = \vec{n}'\varphi = -k_{2x}\varphi\vec{t}$ but this must be a unit vector by lemma 1 so $-k_{2x}\varphi = \pm 1$. We want s_n to increase with s_x so φ is positive,

$$(7)\qquad \varphi = 1/k_{2x} . \quad \text{Thus}$$

$$(8)\qquad k_{1n} = -k_{1x}k_{2x}^{2}\varphi^{3} = -k_{1x}/k_{2x} .$$

The results in equations (6) and (8) and in (3) of section III were obtained by Hostinsky[4].

Given a smooth curve C in elliptic space with its $k_1(s)$ and $k_2(s)$ we have

$$(9) \qquad \int_C k_2\, ds = \int_C \frac{1}{\varphi}\, ds = \int_C \frac{ds_n}{ds_x}\, ds_x = \int_C ds_n = L_n\ ,$$

length of the dual curve as in S^2 , and

$$(10) \qquad \int_C k_{1x}\, ds_x = -\int_C \frac{k_{1n}}{k_{2n}}\, ds_x = -\int_C \frac{k_{1n}}{ds_x/ds_n}\, ds_x = -\int_C k_{1n}\, ds_n\ .$$

In Euclidean space E^3 $\int k_1\, ds \geq 2\pi$ with equality for plane curves, and no simple expression is known for $\int k_2\, ds$.

In S^m we have

$$(1a) \qquad \frac{d}{ds_x}\begin{pmatrix} \vec{x} \\ \vec{x}' \\ \vec{t}_1 \\ \vdots \\ \vec{t}_{m-2} \\ \vec{n} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdot\ \cdot\ \cdot & & 0 \\ -1 & 0 & k_{1x} & 0 & \cdot\ \cdot\ \cdot & 0 \\ \vdots & & & & & \\ 0 & \cdots & -k_{(m-2)x} & 0 & k_{(m-1)x} & \\ 0 & \cdot\ \cdot\ \cdot & & -k_{(m-1)x} & 0 & \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{x}' \\ \vec{t}_1 \\ \vdots \\ \vec{t}_{m-2} \\ \vec{n} \end{pmatrix}$$

If the curve C with frame $(\vec{x}, \vec{x}', \vec{t}_1, \ldots, \vec{t}_{m-2}, \vec{n})$ is not completely contained in a subspace S^k of S^m of dimension $k < m$ we take as its quasi-dual the curve C_n traced by $\vec{n}$ with frame $(\vec{n}, \vec{t}_{m-2}, \ldots, \vec{t}_1, \vec{x}', \vec{x})$. If C is contained in S^k, $k < m$, we consider its quasi-dual in S^k.

Let primes (') and latin numerals denote differentiation with respect to s_x, dots ($^\cdot$) and roman numerals denote differentiation with respect to s_n, and $\varphi = ds_x/ds_n$. Let the symbol $\sim$ denote that only those terms are retained which do not result in linear combinations of previous vectors (and thus would not affect the value of the

determinants to be calculated). Then from (1a)

$$
(11)\quad \begin{cases}
\vec{x}'' = -\vec{x} + k_{1x}\vec{t}_1 \sim k_{1x}\vec{t}_1 \\
\vec{x}''' \sim k_{1x}\vec{t}_1{}' \sim k_{1x}k_{2x}\vec{t}_2 \\
\vdots \\
\vec{x}^{(m-1)} \sim k_{1x} \cdots k_{(m-2)x}\vec{t}_{m-2} \\
\vec{x}^{(m)} \sim k_{1x} \cdots k_{(m-1)x}\vec{n}
\end{cases}
$$

and from (11) and $[\vec{x}, \vec{x}', \vec{t}_1, \ldots, \vec{t}_{m-2}, \vec{n}] = +1$ we get

$$
(12)\quad \begin{cases}
[\vec{x}, \vec{x}', \ldots, \vec{x}^{(m)}] = k_{1x}{}^{m-1}\, k_{2x}{}^{m-2} \cdots k_{(m-1)x} \\
[\vec{x}, \vec{x}', \ldots, \vec{x}^{(m-1)}, \vec{n}] = k_{1x}{}^{m-2}\, k_{2x}{}^{m-3} \cdots k_{(m-2)x} \\
[\vec{x}, \vec{x}', \ldots, \vec{x}^{(m-2)}, \vec{t}_{m-2}, \vec{n}] = k_{1x}{}^{m-3} \cdots k_{(m-3)x} \\
\qquad\vdots \\
[\vec{x}, \vec{x}', \vec{x}'', \vec{t}_2, \ldots, \vec{t}_{m-2}, \vec{n}] = k_{1x}
\end{cases}
$$

Corresponding to equations (11) we have for the dual curve

$$
(11a)\quad \begin{cases}
\vec{n}^{\cdot} = -k_{(m-1)x}\vec{t}_{m-2}\,\varphi \\
\vec{n}^{\cdot\cdot} \sim (-1)^2\, k_{(m-1)x}k_{(m-2)x}\vec{t}_{m-3}\,\varphi^2 \\
\vdots \\
\vec{n}^{(\mu-2)} \sim (-1)^{m-2}\, k_{(m-1)x} \cdots k_{2x}\vec{t}_1\,\varphi^{m-2} \\
\vec{n}^{(\mu-1)} \sim (-1)^{m-1}\, k_{(m-1)x} \cdots k_{1x}\vec{x}'\,\varphi^{m-1} \\
\vec{n}^{(\mu)} \sim (-1)^{m}\, k_{(m-1)x} \cdots k_{1x}\vec{x}\,\varphi^{m}
\end{cases}
$$

and corresponding to equations (12) we have

$$
(12a)\quad \begin{cases}
[\vec{n}, \vec{n}^{\cdot}, \ldots, \vec{n}^{(\mu)}] = k_{1n}{}^{m-1} \cdots k_{(m-1)n} \\
[\vec{n}, \vec{n}^{\cdot}, \ldots, \vec{n}^{(\mu-1)}, \vec{x}] = k_{1n}{}^{m-2} \cdots k_{(m-2)n} \\
[\vec{n}, \ldots, \vec{n}^{(\mu-2)}, \vec{x}', \vec{x}] = k_{1n}{}^{m-3} \cdots k_{(m-3)n} \\
[\vec{n}, \ldots, \vec{n}^{(\mu-3)}, \vec{t}_1, \vec{x}', \vec{x}] = k_{1n}{}^{m-4} \cdots k_{(m-4)n} \\
\qquad\vdots \\
[\vec{n}, \vec{n}^{\cdot}, \vec{n}^{\cdot\cdot}, \vec{t}_{m-4}, \ldots, \vec{t}_1, \vec{x}', \vec{x}] = k_{1n}
\end{cases}
$$

Combining (11a), (12), and (12a) and noting that

$[\vec{n}, \vec{t}_{m-2}, \ldots, \vec{t}_1, \vec{x}', \vec{x}] = \varepsilon_m [\vec{x}, \vec{x}', \vec{t}_1, \ldots, \vec{t}_{m-2}, \vec{n}]$

where $\varepsilon_m = (-1)^{\frac{m(m+1)}{2}}$ we find

$$
(13)\quad \begin{cases}
k_{1n} = \varepsilon_m (-1)^3\, k_{(m-1)x}^{\,2}\, k_{(m-2)x}\, \varphi^3 \\
k_{1n}^{\,2}\, k_{2n} = \varepsilon_m (-1)^6\, k_{(m-1)x}^{\,3}\, k_{(m-2)x}^{\,2}\, k_{(m-3)x}\, \varphi^6 \\
\quad\vdots \\
k_{1n}^{\,s}\, k_{2n}^{\,s-1} \cdots k_{sn} = \varepsilon_m \varepsilon_{s+1} k_{(m-1)x}^{\,s+1} \cdots k_{(m-s-1)x}\, \varphi^{\frac{(s+1)(s+2)}{2}} \\
\qquad\qquad\qquad \text{for } s = 1, \ldots, m-2 \\
k_{1n}^{\,m-1} \cdots k_{(m-1)n} = \varepsilon_m \varepsilon_m\, k_{(m-1)x}^{\,m} \cdots k_{1x}^{\,2}\, \varphi^{m(m+1)/2}
\end{cases}
$$

The argument leading to (7) holds in m-dimensions so

$$(14)\qquad \varphi = 1/k_{(m-1)x}\,.$$

Assuming all $k_{sx} \neq 0$ we compute from (13) and (14)

$$(15)\qquad k_{1n} = \varepsilon_m (-1)^3 \frac{k_{(m-1)x}^{\,2}\, k_{(m-2)x}}{k_{(m-1)x}^{\,3}} = -\varepsilon_m \frac{k_{(m-2)x}}{k_{(m-1)x}}$$

$$k_{2n} = \frac{1}{k_{1n}^{\,2}}\, \varepsilon_m (-1)^6\, k_{(m-1)x}^{\,3}\, k_{(m-2)x}^{\,2}\, k_{(m-3)x}\, \varphi^6$$

$$= \varepsilon_m \frac{k_{(m-1)x}^{\,2}}{k_{(m-2)x}^{\,2}} \quad \frac{k_{(m-1)x}^{\,3}\, k_{(m-2)x}^{\,2}\, k_{(m-3)x}}{k_{(m-1)x}^{\,6}}\,,$$

$$(16)\qquad k_{2n} = \varepsilon_m \frac{k_{(m-3)x}}{k_{(m-1)x}}$$

$$k_{3n} = \varepsilon_m \varepsilon_4 \frac{k_{(m-1)x}^{\,4}\, k_{(m-2)x}^{\,3}\, k_{(m-3)x}^{\,2}\, k_{(m-4)x}}{k_{2n}^{\,2}\, k_{1n}^{\,3}\, k_{(m-1)x}^{\,10}}$$

$$= \varepsilon_m \frac{k_{(m-1)x}^{\,4}\, k_{(m-2)x}^{\,3}\, k_{(m-3)x}^{\,2}\, k_{(m-4)x}\, k_{(m-1)x}^{\,5}}{(-\varepsilon_m)^3\, k_{(m-2)x}^{\,3}\, (\varepsilon_m)^2\, k_{(m-3)x}^{\,2}\, k_{(m-1)x}^{\,10}}$$

$$= \varepsilon_m \frac{k_{(m-4)x}}{k_{(m-1)x}} \quad (-1)^3\, \varepsilon_m^{\,5} \quad = -\frac{k_{(m-4)x}}{k_{(m-1)x}}$$

Assume $k_{jn} = - k_{(m-j-1)x}/k_{(m-1)x}$ for $j = 3, \ldots, s-1$. Then

$$k_{sn} = \varepsilon_m \varepsilon_{s+1} \frac{k_{(m-1)x}^{s+1} \, k_{(m-2)x}^{s} \ldots k_{(m-s-1)x}}{k_{1n}^{s} \ldots k_{(s-1)n}^{2} \, k_{(m-1)x}^{(s+1)(s+2)/2}}$$

$$= \frac{\varepsilon_m \, \varepsilon_{s+1} \, k_{(m-s-1)x} \, k_{(m-1)x}^{s+1+s+\ldots+2}}{(-\varepsilon_m)^{s} \, \varepsilon_m^{s-1} \, (-1)^{s-2} \ldots (-1)^{2} k_{(m-1)x}^{(s+1)(s+2)/2}}$$

$$= \varepsilon_{s+1} (\varepsilon_m)^{1-s-(s-1)} \, k_{(m-s-1)x} \, k_{(m-1)x}^{-1} \, (-1)^{s(s-3)/2 - s}$$

$$= \frac{k_{(m-s-1)x}}{k_{(m-1)x}} \ \text{sign } k_{sn}$$

where $\text{sign } k_{sn} = (-1)^{-s+s(s-3)/2} \varepsilon_{s+1}$

$$= (-1)^{s(s-3)/2 \, + \, (s+1)(s+2)/2 \, - \, s}$$
$$= (-1)^{s^2-s+1} = -1$$

Hence by induction

(17) $\quad k_{sn} = - k_{(m-s-1)x}/k_{(m-1)x} \quad , \; s = 3, \ldots, m-2$

For $s = m-1$ this formula does not apply. By (6) and equation (3) of section III we have $k_{1n} = 1/k_{1x}$ and $k_{2n} = 1/k_{2x}$ for $m = 2$ and $m = 3$ respectively, For $m > 3$ we have from the last of equations (13)

$$k_{(m-1)n} = \frac{k_{(m-1)x}^{m} \, k_{(m-2)x}^{m-1} \ldots k_{1x}^{2} \, \varphi^{m(m+1)/2}}{k_{1n}^{m-1} \, k_{2n}^{m-2} \ldots k_{(m-2)n}^{2}}$$

$$= \frac{k_{(m-2)x}^{m-1} \ldots k_{1x}^{2} \, k_{(m-1)x}^{m+(m-1)+\ldots+2 \, - \, m(m+1)/2}}{(-\varepsilon_m)^{m-1} k_{(m-2)x}^{m-1} (\varepsilon_m)^{m-2} k_{(m-3)x}^{m-2} (-1)^{m-3} k_{(m-4)x}^{m-3} \ldots (-1)^{2} k_{1x}^{2}}$$

$$= \frac{\text{sign } k_{(m-1)n}}{k_{(m-1)x}}$$

where $\operatorname{sign} k_{(m-1)n} = (\varepsilon_m)^{3-2m}/(-1)^{m-1+(m-3)+\ldots+2}$

$$= (-1)^{3m(m+1)/2 - (m-1)(m-4)/2 - m + 1}$$

$$= (-1)^{m^2+3m-1} = -1 \quad \text{so}$$

(18) $\qquad k_{(m-1)n} = -1/k_{(m-1)x} \quad , \quad m \geq 3.$

Equations (15) through (18) give all the generalized curvature quantities for the quasi-dual curve C_n.

V Duality for Curves and Surfaces in S^3

We first find the dual of a curve C in S^3 given together with its frame and Frenet matrix

$$(1) \qquad \frac{d}{ds}\begin{pmatrix}\vec{x}\\ \vec{x}'\\ \vec{e}_3\\ \vec{n}\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ -1 & 0 & k_1 & 0\\ 0 & -k_1 & 0 & k_2\\ 0 & 0 & -k_2 & 0\end{pmatrix}\begin{pmatrix}\vec{x}\\ \vec{x}'\\ \vec{e}_3\\ \vec{n}\end{pmatrix}$$

where $[\vec{x}, \vec{x}', \vec{e}_3, \vec{n}] = 1$.

If we think of C as a class curve, the envelope of its tangent vectors $\vec{x}'$ ($\vec{x}'$ is the intersection with the unit sphere in E^4 of the plane of $\vec{x}$ and $\vec{x}'$), its dual will be the developable surface S generated by the dual lines which are the intersection with the unit sphere in E^4 of the planes of $\vec{e}_3$ and $\vec{n}$. Whereas $\vec{x}$ represents the point coordinates of curve C in E^4, $\vec{e}_3\wedge\vec{n}$ represents the line coordinates in the Grassman manifold $G_{2,2}$ (see Auslander [5] p 176) of the rulings of S.

Calculating from (1) by the rules of exterior calculus, as in the example

$$\frac{d}{ds}(\vec{e}_3\wedge\vec{n}) = \frac{d}{ds}\vec{e}_3 \wedge\vec{n} + \vec{e}_3\wedge \frac{d}{ds}\vec{n}$$

$$= (-k_1\vec{x}' + k_2\vec{n})\wedge\vec{n} + \vec{e}_3\wedge(-k_2\vec{e}_3) = -k_1\vec{x}'\wedge\vec{n} ,$$

we obtain for the surface S dual to C

$$(2) \qquad \frac{d}{ds}\begin{pmatrix}\vec{e}_3\wedge\vec{n}\\ \vec{n}\wedge\vec{x}\\ \vec{x}\wedge\vec{x}'\\ \vec{x}'\wedge\vec{e}_3\\ \vec{x}\wedge\vec{e}_3\\ \vec{x}'\wedge\vec{n}\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & -k_1\\ 0 & 0 & 0 & 0 & k_2 & -1\\ 0 & 0 & 0 & 0 & k_1 & 0\\ 0 & 0 & 0 & 0 & -1 & k_2\\ 0 & -k_2 & -k_1 & 1 & 0 & 0\\ k_1 & 1 & 0 & -k_2 & 0 & 0\end{pmatrix}\begin{pmatrix}\vec{e}_3\wedge\vec{n}\\ \vec{n}\wedge\vec{x}\\ \vec{x}\wedge\vec{x}'\\ \vec{x}'\wedge\vec{e}_3\\ \vec{x}\wedge\vec{e}_3\\ \vec{x}'\wedge\vec{n}\end{pmatrix}$$

Given a surface S in S^3 we have two cases. If S is a developable surface the tangent plane α at a point O is tangent at every point of the ruling through O. The pole P of α is the dual point corresponding to every point of the ruling. The points dual to the rulings generate a curve C dual to the developable surface S.

If S is not developable, let S be given by $\vec{x} = \vec{x}(u,v)$. Take the unit surface normal $\vec{n}$ and unit vectors $\vec{e}_2$, $\vec{e}_3$ in the tangent plane of S such that $\vec{x}$, $\vec{e}_2$, $\vec{e}_3$, $\vec{n}$ form an orthonormal frame and $[\vec{x}, \vec{e}_2, \vec{e}_3, \vec{n}] = +1$.

We have given

$$(3) \qquad d\begin{pmatrix} \vec{x} \\ \vec{e}_2 \\ \vec{e}_3 \\ \vec{n} \end{pmatrix} = \begin{pmatrix} 0 & \omega_1 & \omega_2 & 0 \\ -\omega_1 & 0 & \omega_{23} & \omega_{24} \\ -\omega_2 & -\omega_{23} & 0 & \omega_{34} \\ 0 & -\omega_{24} & -\omega_{34} & 0 \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{e}_2 \\ \vec{e}_3 \\ \vec{n} \end{pmatrix}$$

so $d\vec{x} = \omega_1\vec{e}_2 + \omega_2\vec{e}_3$.

By Poincaré's formula (Guggenheimer [6] p 190) $dd\vec{x} = 0$ so by (3) and exterior differentiation

$$0 = d\omega_1\vec{e}_2 + \omega_1\wedge(-\omega_1\vec{x} + \omega_{23}\vec{e}_3 + \omega_{24}\vec{n})$$
$$+ d\omega_2\vec{e}_3 + \omega_2\wedge(-\omega_2\vec{x} - \omega_{23}\vec{e}_2 + \omega_{34}\vec{n}) .$$

But $\vec{x}$, $\vec{e}_2$, $\vec{e}_3$, $\vec{n}$ are linearly independent so their coefficients must vanish, in particular for $\vec{n}$, so

$$\omega_1\wedge\omega_{24} + \omega_2\wedge\omega_{34} = 0.$$

Then by Cartan's theorem (Guggenheimer [6] p 189) there is a frame $\vec{x}$, $\vec{e}_2{}^*$, $\vec{e}_3{}^*$, $\vec{n}$ such that the second fundamental form $II = d\vec{x}.d\vec{n}$ is diagonal, i.e.

$$(4)\qquad d\begin{pmatrix}\vec{x}\\ \vec{e}_2{}^*\\ \vec{e}_3{}^*\\ \vec{n}\end{pmatrix} = \begin{pmatrix}0 & \omega_1 & \omega_2 & 0\\ -\omega_1 & 0 & \omega_{23} & -k_1\omega_1\\ -\omega_2 & -\omega_{23} & 0 & -k_2\omega_2\\ 0 & k_1\omega_1 & k_2\omega_2 & 0\end{pmatrix}\begin{pmatrix}\vec{x}\\ \vec{e}_2{}^*\\ \vec{e}_3{}^*\\ \vec{n}\end{pmatrix}.$$

The invariants k_1, k_2 are the principal curvatures of the surface S and are related to the Gauss curvature, K, and mean curvature, H, of the surface by

$$K = k_1k_2$$
$$H = \tfrac{1}{2}(k_1 + k_2).$$

From (4) the fundamental forms of the surface S are

$$I = d\vec{x}.d\vec{x} = \omega_1{}^2 + \omega_2{}^2$$
$$II = d\vec{x}.d\vec{n} = k_1\omega_1{}^2 + k_2\omega_2{}^2$$
$$III = d\vec{n}.d\vec{n} = k_1{}^2\omega_1{}^2 + k_2{}^2\omega_2{}^2$$

so we have the relation

$$k_1k_2(\omega_1{}^2 + \omega_2{}^2) - (k_1+k_2)(k_1\omega_1{}^2 + k_2\omega_2{}^2) + (k_1{}^2\omega_1{}^2 + k_2{}^2\omega_2{}^2)$$
$$(5)\qquad = K\ I - 2H\ II + III = 0$$

Now the direction dual to hyperplane $\begin{pmatrix}\vec{x}\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix}$ in E^4 is $\vec{n}$. From (3) $d\vec{n} = -\omega_{24}\vec{e}_2 - \omega_{34}\vec{e}_3$ and this must be the tangent plane in S^3 of the dual S_n to surface S. So S_n has normal direction $\vec{x}$ and frame $\begin{pmatrix}\vec{n}\\ \vec{e}_3\\ \vec{e}_2\\ \vec{x}\end{pmatrix}$.

Thus from (3) $I_n = III$, $II_n = II$, $III_n = I$ and by equation (5)

$$K_nI_n - 2H_nII_n + III_n = K_n\ III - 2H_n\ II + I = 0$$

so if $K \neq 0$, $K_n = 1/K$ and $H_n = H/K$ are the Gauss and mean curvatures of the dual surface S_n.

For S_n we have corresponding to (4)

$$d \begin{pmatrix} \vec{n} \\ \vec{e}_3{}^* \\ \vec{e}_2{}^* \\ \vec{x} \end{pmatrix} = \begin{pmatrix} 0 & \omega_1{}^* & \omega_2{}^* & 0 \\ -\omega_1{}^* & 0 & \omega_{23}{}^* & -k_{1n}\omega_1{}^* \\ -\omega_2{}^* & -\omega_{23}{}^* & 0 & -k_{2n}\omega_2{}^* \\ 0 & k_{1n}\omega_1{}^* & k_{2n}\omega_2{}^* & 0 \end{pmatrix} \begin{pmatrix} \vec{n} \\ \vec{e}_3{}^* \\ \vec{e}_2{}^* \\ \vec{x} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & k_2\omega_2 & k_1\omega_1 & 0 \\ -k_2\omega_2 & 0 & -\omega_{23} & -\omega_2 \\ -k_1\omega_1 & \omega_{23} & 0 & -\omega_1 \\ 0 & \omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} \vec{n} \\ \vec{e}_3{}^* \\ \vec{e}_2{}^* \\ \vec{x} \end{pmatrix}$$

so $$k_{1n} = \frac{k_{1n}\omega_1{}^*}{\omega_1{}^*} = \frac{\omega_2}{k_2\omega_2} = \frac{1}{k_2}$$

$$k_{2n} = 1/k_1$$

are the principal curvatures of the dual surface S_n.

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